31. Energy & Power Spectral Densities

Prof. Mohammed Hawa
Electrical Engineering
The University of Jordan

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$S_x(\omega) = \mathcal{F}\{R_{xx}(\tau)\}$$

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega) \, d\omega$$

Reminder:

If we are given the equation for a time-domain signal $x(t)$, which is voltage or current, we can calculate **instantaneous power** $p(t) = x^2(t)$, as well as the **total energy** [Joule] in the signal $x(t)$,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

and the **average power** [Watt] in the signal $x(t)$,

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

How about if we are given the frequency-domain $X(\omega) = \mathcal{F}\{x(t)\}$?

The **energy spectral density (ESD)** $\Psi_x(\omega)$ of a signal $x(t)$ is

$$\Psi_x(\omega) = |X(\omega)|^2$$

The ESD describes the relative amount of energy of the signal $x(t)$ versus frequency ω. Parseval's theorem tells us that the total area under the ESD (divided by 2π) is the total energy in the signal $x(t)$.

Parseval's theorem states that the **total energy** E_x in the signal $x(t)$ can be calculated either from time-domain or frequency-domain as follows:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_x(\omega) \, d\omega$$

Q1. Determine the energy spectral density $\Psi_x(\omega)$ for the signal $x(t) = A \operatorname{rect}\left(\frac{t}{\tau}\right)$, then find the signal total energy E_x using both time-domain and frequency-domain.

Q1. Solution. We found earlier that for $x(t) = A \operatorname{rect}\left(\frac{t}{\tau}\right)$

$$X(\omega) = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Hence,

$$\Psi_x(\omega) = |X(\omega)|^2 = \left|A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)\right|^2 = A^2\tau^2 \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

From Parseval's theorem, we can calculate total energy E_x from:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E_x = \int_{-\tau/2}^{\tau/2} A^2 \, dt = A^2 \, [t]_{-\tau/2}^{\tau/2} = A^2\tau$$

which is the area under the squared rectangle. From frequency-domain:

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_x(\omega) \, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} A^2\tau^2 \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right) \, d\omega = \frac{A^2\tau^2}{2\pi}\int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right) \, d\omega$$

It is well-known that the integral of *square* of normalized sinc( ) is:

$$\int_{-\infty}^{\infty} \frac{\sin^2(\pi\gamma)}{(\pi\gamma)^2} \, d\gamma = \int_{-\infty}^{\infty} \operatorname{sinc}^2(\gamma) \, d\gamma = 1$$

Bu substituting $\gamma = \frac{\omega\tau}{2\pi}$ and $d\gamma = \frac{\tau}{2\pi} d\omega$, we get

$$\frac{\tau}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right) \, d\omega = 1$$

$$\frac{\tau}{2\pi}\int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)\, d\omega = 1$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right) d\omega = \frac{2\pi}{\tau}$$

This means,

$$E_x = \frac{A^2\tau^2}{2\pi}\int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)\, d\omega = \frac{A^2\tau^2}{2\pi} \times \frac{2\pi}{\tau} = A^2\tau$$

The **power spectral density (PSD)** $S_x(\omega)$ of a signal $x(t)$ is

$$S_x(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} = \mathcal{F}\{R_{xx}(\tau)\}$$

The PSD is a function that describes the relative amount of power of a given signal versus frequency. Power signals (periodic or aperiodic) have a PSD not an ESD (their ESD is infinite).

Some $S_x(\omega)$ properties:

$S_x(\omega) \geq 0$; and $S_x(\omega)$ is always real-valued

$S_x(-\omega) = S_x(\omega)$, even symmetry

The total area under the PSD (divided by 2π) is the average power in the signal $x(t)$:

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega = R_{xx}(0)$$

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

The PSD of periodic signal consist of a group of impulses, while the PSD of aperiodic signal is a smooth continuous curve. This is because periodic signals are actually the sum of an infinite number of sinusoids.

For periodic signals, we can also obtain average power from Fourier series coefficients of the signal. Since the harmonics are orthogonal signals, the power of the harmonics can be added using superposition

From the compact Fourier series $x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n)$

$$P_x = \left(\frac{c_0}{2}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{c_n^2}{2}\right)$$

This is another form of Parseval's theorem, as applied to power signals. For the complex exponential Fourier series, and real-valued signal $x(t)$

$$P_x = \left(\frac{c_0}{2}\right)^2 + 2\sum_{n=1}^{\infty}\left(\frac{c_n}{2}\right)^2 = |\alpha_0|^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=-\infty}^{-1}|\alpha_n|^2 = \sum_{n=-\infty}^{\infty}|\alpha_n|^2$$

Q2. Determine and sketch the power spectral density $S_x(\omega)$ for the signal $x(t) = A\cos(\omega_0 t)$, then find the signal average power P_x from both time-domain and frequency-domain.

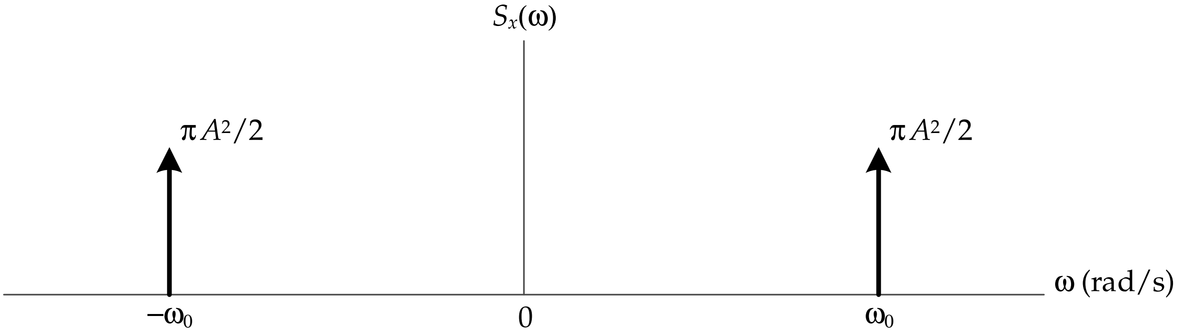
Q2. Solution. From an earlier example, we found that

$$R_{xx}(\tau) = \frac{A^2}{2}\cos(\omega_0 \tau)$$

Hence,

$$S_x(\omega) = \mathcal{F}\{R_{xx}(\tau)\} = \mathcal{F}\left\{\frac{A^2}{2}\cos(\omega_0 \tau)\right\}$$

$$S_x(\omega) = \pi\frac{A^2}{2}\,\delta(\omega-\omega_0) + \pi\frac{A^2}{2}\,\delta(\omega+\omega_0)$$



$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \frac{1}{2\pi}\left[\pi\frac{A^2}{2} + \pi\frac{A^2}{2}\right] = \frac{A^2}{2}$$

Or,

$$P_x = R_{xx}(0) = \frac{A^2}{2}\cos(\omega_0\tau)\big|_{\tau=0} = \frac{A^2}{2}\cos(0) = \frac{A^2}{2}$$